Fibonacci Day maths activities for IGCSE and GCSE: SL / AS: answers

IGCSE Math Revision · IB Maths SL and HL (AA and AI) · 23 November · Teacher answers and notes

The starter is non-calculator. A GDC helps in tasks A and B.

Teacher notes

- Fibonacci Day is 23 November because 11/23, written month first, reads 1, 1, 2, 3.
- In task A (b), insist on the reason for rejecting $(1 - \sqrt{5})/2$: every ratio is positive.
- Task B (d) makes a good display: students draw the squares on squared paper and add the arcs with compasses.
- Extension 1 explains task A: $|\psi| < 1$, so $\psi^n \to 0$ and $F_{n+1}/F_n \to \varphi$.

Starter

1. **2, 3, 5, 8, 13, 21, 34, 55**
   - Add the two previous terms each time: 1 + 1 = 2, 1 + 2 = 3, 2 + 3 = 5, and so on up to 21 + 34 = 55.
2. **$x = (1 + \sqrt{5})/2$ or $(1 - \sqrt{5})/2$**
   - $x^2 - x - 1 = 0$, so $x = (1 \pm \sqrt{(1 + 4)})/2 = (1 \pm \sqrt{5})/2$.
3. **20, which is $F_8 - 1$**
   - 1 + 1 + 2 + 3 + 5 + 8 = 20
   - $F_8 = 21$, so the sum is one less than F_8.
4. **1.615**
   - 21 ÷ 13 = 1.6153...

Task A: The limit of the ratios

a. **1, 2, 1.5, 1.667, 1.6, 1.625, 1.615, 1.619: the ratios settle down, alternately above and below about 1.618**
   - 1/1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13, 34/21
   - $r_8 = 34 \div 21 = 1.619$ (3 d.p.)

b. **$\varphi = (1 + \sqrt{5})/2 \approx 1.618$**
   - $F_{n+2}/F_{n+1} = 1 + F_n/F_{n+1}$, that is $r_{n+1} = 1 + 1/r_n$.
   - In the limit $\varphi = 1 + 1/\varphi$, so $\varphi^2 = \varphi + 1$ and $\varphi = (1 \pm \sqrt{5})/2$.
   - Every ratio is positive, so $\varphi = (1 + \sqrt{5})/2 = 1.618$ (3 d.p.).

c. **$\varphi^3 = 2\varphi + 1$, $\varphi^4 = 3\varphi + 2$: Fibonacci numbers again**
   - $\varphi^3 = \varphi \times \varphi^2 = \varphi^2 + \varphi = (\varphi + 1) + \varphi = 2\varphi + 1$.
   - $\varphi^4 = \varphi \times \varphi^3 = 2\varphi^2 + \varphi = 2(\varphi + 1) + \varphi = 3\varphi + 2$.
   - In general $\varphi^n = F_n\varphi + F_{n-1}$.

Task B: Golden rectangles and the spiral

a. **$1 \div (\varphi - 1) = \varphi$, because $\varphi(\varphi - 1) = 1$**
   - $\varphi(\varphi - 1) = \varphi^2 - \varphi = (\varphi + 1) - \varphi = 1$.
   - So $1 \div (\varphi - 1) = \varphi$: the long side is φ times the short side.

b. **48.5 cm**
   - $30\varphi = 30 \times 1.6180... = 48.54...$
   - 48.5 cm (3 s.f.)

c. **$273\ \text{cm}^2 = 13 \times 21$**
   - 1 + 1 + 4 + 9 + 25 + 64 + 169 = 273
   - $13 \times 21 = 273$. In general $F_1^2 + ... + F_n^2 = F_nF_{n+1}$.

d. **$33\pi/2 \approx 51.8$ cm**
   - Each quarter circle has length $\frac{1}{4} \times 2\pi r = \pi r/2$.
   - Total $= \pi/2 \times (1 + 1 + 2 + 3 + 5 + 8 + 13) = \pi/2 \times 33 = 33\pi/2 = 51.83...$

Task C: Fibonacci identities

a. **143 = 144 − 1**
   - 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 = 143
   - $F_{11} = 89$ and $F_{12} = 144$, so $F_{12} - 1 = 143$.

b. **1, −1, 1, −1, 1: the result is $(-1)^n$**
   - $n = 2$: $1 \times 2 - 1 = 1$. $n = 3$: $1 \times 3 - 4 = -1$. $n = 4$: $2 \times 5 - 9 = 1$.
   - $n = 5$: $3 \times 8 - 25 = -1$. $n = 6$: $5 \times 13 - 64 = 1$.
   - So $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$.

c. **10**
   - The pattern of odd (O) and even (E) is O, O, E, O, O, E, ..., because O + O = E, O + E = O and E + O = O.
   - So F_n is even exactly when n is a multiple of 3: n = 3, 6, ..., 30, which is 10 numbers.

Extension

1. **$F_{20} = 6765$**
   - $n = 1$: $(\varphi - \psi)/\sqrt{5} = \sqrt{5}/\sqrt{5} = 1$.
   - $n = 2$: $(\varphi^2 - \psi^2)/\sqrt{5} = (\varphi - \psi)(\varphi + \psi)/\sqrt{5} = \sqrt{5} \times 1/\sqrt{5} = 1$.
   - $n = 20$: $(\varphi^{20} - \psi^{20})/\sqrt{5} = 6765$.
2. **Proof**
   - Base case: $F_1 = 1 = F_3 - 1$.
   - Assume the sum of the first k terms is $F_{k+2} - 1$. Adding F_{k+1} gives $F_{k+2} + F_{k+1} - 1 = F_{k+3} - 1$.
   - True for $n = 1$, and true for $n = k + 1$ whenever it is true for $n = k$, so true for every positive integer n.

Every question is original and every answer was checked before publishing.